Evaluating the CDF for m weighted sums of n correlated lognormal random variables

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We show that one can evaluate the cumulative probability density function of m weighted sums of n correlated lognormal variables with Monte Carlo simulation rapidly, by deriving its joint probability density function. The adaptive Monte Carlo method allows us to estimate the number of rounds required to achieve a given tolerance. The need for evaluating this function rapidly occurs in many applications, for instance for pricing combinatorial options for bandwidth markets.

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Abstract

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1 Introduction

1.1 Background

Systems which state depend on very many independent factors can, due to the Central Limit Theorem, often be modeled as changing state with normal distributed random steps. If this modeling works for sufficiently small time scales, the system can be modeled as being driven by the continuous time analog, the Wiener process. If the state change is proportional to the current state, such a system generates lognormal trajectories, i.e. the future state of the system follows a lognormal distribution.

In our prior work on load balancing of computer network, sums of correlated lognormals arise as modeling the cost of a routed path in the computer network. In a prior bandwidth market simulation, the short time capacity cost of each router was found to be approximately lognormal [10], and therefore complex network services such as virtual channels, etc., can be described as derivative contracts on router capacity. To evaluate an efficient price for a virtual channel

in a routed network, one needs to evaluate the CDF for sums of lognormal variables in an efficient way [9]. In the standard applications of mathematical finance one has usually priced assets which are to a large extent uncorrelated (stocks or interest rate derivatives), but in bandwidth markets, the underlying network structure makes capacity prices correlated, and hence the need to also handle correlated lognormal variables.

The high dimensionality (due to the number of network routers) makes standard methods from mathematical finance fail. For instance, the standard binomial tree method [3] requires exponentially more memory as the number of assets grow. Therefore we will have to rely on the Monte Carlo method to evaluate the CDF within a probabilistically bounded error.

Other applications which involve the evaluation the expected value of a discontinuous function of the future state are interference in cellular phone networks [6], or so called quality options [2]. In these, as well in our application one needs to compute the cumulative density function (CDF) for a weighted sum of lognormal variables within a specified tolerance.

1.2 Weighted sums of lognormal variables

We give a formula for the PDF of m weighted sums of correlated lognormals that arise from n correlated processes driven by n independent Wiener processes, $(W(t+\tau)-W(t))/\sqrt{\tau} \in N(0,1)$ in terms of an n dimensional integral over $\mathcal{R}^n$.

No closed form solution of the CDF for weighted sums of lognormals has been found in the literature. Beaulieu et al. [1], dismiss the idea of determining the PDF of a sum by computing the inverse Fourier transform of the product of the Fourier transforms of the PDFs of the summands, on the grounds that the Fourier transform for lognormal variables is not known, and that numerical Fourier transform is difficult due to highly oscillating integrands with slowly decaying tails. Instead, they suggest approximating the sum of independent lognormals with another lognormal density function. This is a crude approximation, since, as they are aware, the sum of lognormals is *not* a lognormal distribution itself, nor does the method provide much insight into the size of the approximation error.

Although a promising paper by Leipnik [5] giving the characteristic function of the lognormal PDF (albeit in a form shown to be difficult to evaluate numerically) appeared to be a first step toward the solution, its promised follow-up paper has not yet been published.

Milevsky *et al.* [7] approximate an unweighted finite sum of correlated lognormal variables with an infinite unweighted sum of lognormals, and find that the inverse of the infinite sum is gamma distributed. They approximate the inverse finite sum with a gamma distribution by moment matching.

Boyle et al. approximate the minimum of several single lognormal assets S_i as

$$\begin{aligned} E[\min(S_1, ..., S_n)] &= E[\exp(\max(-\log S_1, ..., -\log S_n))] \\ &\approx E[\exp(-V_n)] \\ &\approx \exp(-\mu)\left[1 + \frac{\mu_2}{2!} - \frac{\mu_3}{3!} + \frac{\mu_4}{4!}\right] \end{aligned}$$

where $V_i = \max(V_{i-1}, \log S_i)$ is approximated with a normal distribution with suitable moments μ_k. This is an approximation to exact formulas provided by Johnson [4] which prove to be very complex to evaluate for more than two assets.

The difficulty to find closed form solutions to these problems suggests that the general problem of weighted sums of lognormals also has to be tackled by numerical means.

We derive the PDF for the weighted sum of lognormals, and use the Monte Carlo method to evaluate the CDF to within an estimated error tolerance.

2 The density functions

2.1 The PDF for a correlated multi-dimensional lognormal process X

Let $W_i(t)$, $i = 1, \ldots, n$ be an n dimensional Wiener process, i.e. $W_i(t) \in N(0, \sqrt{t})$ are i.i.d. Let $X_i(t)$, $i = 1, \ldots, n$ be a correlated lognormal stochastic process

$$dX_i(t) = \sum_{j=1}^{n} X_i(t)\sigma_{ij} dW_j(t)$$

with linearly independent components, which implies that σ^{-1} exists. The n-dimensional normal process $Y_i(t)$ is defined by

$$Y_i(t) = \log X_i(t),\ i = 1, \ldots, n$$

A Taylor series expansion of $dY_i(t)$ to order dt gives

$$\begin{aligned} dY_i(t) &= \frac{\partial \log X_i(t)}{\partial X_i(t)} dX_i(t) + \frac{1}{2}\frac{\partial^2 \log X_i(t)}{(\partial X_i(t))^2}(dX_i(t))^2 + O((dX_i(t))^3) \\ &= \sum_{j=1}^{n} \sigma_{ij} dW_j(t) - \frac{1}{2}(\sum_{j=1}^{n} \sigma_{ij} dW_j(t))^2 + O(\sum_{j,k,l} dW_j(t) dW_k(t) dW_l(t)) \\ &= \sum_{j=1}^{n} \sigma_{ij} dW_j(t) - \frac{1}{2}(\sum_{j=1}^{n} \sigma_{ij}^2) dt + O(\sum_{j} dW_j(t) dt) \end{aligned}$$

since Wiener processes have the property $(dW_i)^2 = dt$. Therefore

$$\begin{aligned} Y_i(t) - Y_i(0) &= \int_0^t dY_i(s) \\ &= \sum_{j=1}^{n} \sigma_{ij} \int_0^t dW_j(s) - \frac{1}{2}(\sum_{j=1}^{n} \sigma_{ij})^2 \int_0^t ds \\ &= \sum_{j=1}^{n} \sigma_{ij} W_j(t) - \frac{1}{2}(\sum_{j=1}^{n} \sigma_{ij}^2) t \end{aligned}$$

and we see that $Y_i(t) \in N(\mu_i, s_i^2)$ where mean $\mu_i = -\frac{1}{2}(\sum_{j=1}^{n} \sigma_{ij}^2)t$, and variance $s_i^2 = t\sum_{j=1}^{n} \sigma_{ij}^2$. The covariance is

$$\begin{aligned} Cov[Y_i(t), Y_j(t)] &= \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} (\sum_{k=1}^{n} \sigma_{ik} w_k - \mu_i)(\sum_{k=1}^{n} \sigma_{jk} w_k - \mu_j) \\ &\qquad \underbrace{\times \frac{1}{(2\pi t)^{n/2}} \exp(-\frac{1}{2t}\sum_{k=1}^{n} w_k^2) dw_1 \cdots dw_n}_{\equiv f(\bar{w}) d\bar{w}} - \mu_i \mu_j \\ &= \mu_i \mu_j \underbrace{\int f(\bar{w}) d\bar{w}}_{=1} - \mu_i \sum_{k=1}^{n} \sigma_{jk} \underbrace{\int w_k f(\bar{w}) d\bar{w}}_{=E[W_k]=0} - \mu_j \sum_{k=1}^{n} \sigma_{ik} \underbrace{\int w_k f(\bar{w}) d\bar{w}}_{=0} \\ &\qquad + \int (\sum_{k=1}^{n} \sigma_{ik} w_k)(\sum_{k=1}^{n} \sigma_{jk} w_k) f(\bar{w}) d\bar{w} - \mu_i \mu_j \\ &= \sum_{k=1}^{n} \sigma_{ik} \sigma_{jk} \underbrace{\int w_k^2 f(\bar{w}) d\bar{w}}_{=Var[W_k(t)]=t} + \sum_{k \neq l} \sigma_{ik} \sigma_{jl} \underbrace{\int w_k w_l f(\bar{w}) d\bar{w}}_{=Cov[W_k(t), W_l(t)]=0} \end{aligned}$$

$$= \quad t\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk}$$

where we let $\bar{w}$ denote the vector $(w_1, ..., w_n)^T$ and let $d\bar{w}$ denote the volume element $dw_1 \cdots dw_n$. Hence, if we denote $\mathbf{C} = \sigma\sigma^T$, the covariance matrix is $t\,\mathbf{C}$, and the mean is $\mu_i = -\frac{1}{2}t\,C_{ii}$. The joint density function for correlated normal variables is

$$p_{\bar{Y}}(\bar{y})dy_1 \cdots dy_n = \frac{1}{(2\pi t)^{n/2}\sqrt{\det(\mathbf{C})}}\exp\left(-\frac{1}{2t}(\bar{y}-\bar{\mu})^T\mathbf{C}^{-1}(\bar{y}-\bar{\mu})\right)dy_1 \cdots dy_n$$

so by using $Y_i = \log X_i$ and performing the coordinate change

$$\begin{aligned} y_i &= \log x_i \\ dy_i &= \frac{1}{x_i}dx_i \end{aligned}$$

we obtain

$$\begin{aligned} p_{\bar{X}}(\bar{x}) &= \frac{1}{(2\pi t)^{n/2}\sqrt{\det(\mathbf{C})}}\exp\left(-\frac{1}{2t}(\log\bar{x}-\bar{\mu})^T\mathbf{C}^{-1}(\log\bar{x}-\bar{\mu})\right)\frac{1}{x_1 \cdots x_n} \\ &= p_{\bar{Y}}(\sigma^{-1}(\log\bar{x}-\bar{\mu})) \\ p_{\bar{Y}}(\bar{y}) &= \prod_{i=1}^{n}\phi(y_i) \end{aligned}$$

with the substitution $\bar{Y} = \sigma^{-1}(\log\bar{X} - \bar{\mu})$, and $\log\bar{x} \equiv (\log x_1, ..., \log x_n)^T$, and where $\phi(x)$ is the PDF for the standard normal distribution.

2.2 The PDF for $\bar{S} = \mathbf{W}\bar{X}$

Consider m linearly independent sums of n correlated variables $S_i = \sum_{j=1}^{n} w_{ij}X_j$, or $\bar{S} = \mathbf{W}\bar{X}$. Note that rank$(\mathbf{W}) = m$, and $m \leq n$. The joint density function for the sums $S_1, ..., S_m$ is

$$q_{\bar{S}}(s_1, \ldots, s_m) = \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} dx_1 \cdots dx_n \delta^{(m)}(\mathbf{W}\bar{x} - \bar{s})p_{\bar{X}}(x_1, \ldots . x_n)$$

where $p_{\bar{X}}(\cdot)$ is the joint density function for the correlated variables, and

$$\delta^{(m)}(\mathbf{W}\bar{s} - \bar{k}) \equiv \prod_{i=1}^{m} \delta(\sum_{j=1}^{n} w_{ij}x_j - s_i)$$

where $\delta(x)$ is the Kronecker delta function, defined by

$$\int_a^b f(x)\delta(x)dx = \begin{cases} f(0) & \text{if } a \leq 0 \leq b \\ 0 & \text{otherwise} \end{cases}$$

The m constraints imposed by the delta functions reduce the dimension of the integral by m. To see this, partition $\mathcal{R}^n$ into two orthogonal subspaces, $Ker(\mathbf{W})$ and $Im(\mathbf{W}^T)$. $Ker(L) \equiv \{\bar{x}|L\bar{x} = 0\}$ is the kernel, or null space, of the linear operator L, and $Im(L) = \{\bar{y}|\exists\bar{x} : L\bar{x} = \bar{y}\}$ is the image, or range, of L. They are orthogonal, since

$$\forall\bar{x} \in Ker(\mathbf{W}), \bar{y} \in Im(\mathbf{W}^T) : (\bar{x}, \bar{y}) = (\bar{x}, \mathbf{W}^T\bar{z}) = (\mathbf{W}\bar{x}, \bar{z}) = (0, \bar{z}) = 0$$

where $(\bar{x}, \bar{y})$ denotes the scalar product.

Construct n orthonormal base vectors $\bar{v}_i$, $i = 1, \ldots, n$ such that $\bar{v}_1, \ldots, \bar{v}_m \in Im(\mathbf{W}^T)$ and $\bar{v}_{m+1,\ldots,n} \in Ker(\mathbf{W})$, for instance by first finding the nullspace basis, and then use Gram-Schmidt's method. Let $\mathbf{V}$ denote the $n \times n$ matrix with column i equal to $\bar{v}_i$. Performing the coordinate change

$$\begin{aligned} \bar{x} &= \mathbf{V}\bar{y} \\ d\bar{x} &= \det(\mathbf{V})d\bar{y} = d\bar{y} \end{aligned}$$

gives us

$$\begin{aligned} q_{\bar{S}}(\bar{s}) &= \int d\bar{x}\delta^{(m)}(\mathbf{W}\bar{x} - \bar{s})p_{\bar{X}}(\bar{x}) \\ &= \int d\bar{y}\delta^{(m)}(\mathbf{W}\mathbf{V}\bar{y} - \bar{s})p_{\bar{X}}(\mathbf{V}\bar{y}) \\ &= \underbrace{\int \cdots \int}_{n-m} dy_{m+1} \cdots dy_n \underbrace{\int \cdots \int}_{m} dy_1 \cdots dy_m \prod_{i=1}^{m} \delta(\sum_{j=1}^{m}\{\mathbf{W}\mathbf{V}\}_{ij}y_j - s_i)p_{\bar{X}}(\mathbf{V}\bar{y}) \end{aligned}$$

Let $\mathbf{M}$ be the linear operator

$$\mathbf{M}\bar{y} = \sum_{j=1}^{m} \{\mathbf{WV}\}_{ij} y_j$$

which is the "leftmost" $m \times m$ sub-matrix of $\mathbf{WV}$. $\mathbf{M}$ is invertible since it maps $\mathcal{R}^m \to \mathcal{R}^m$. Performing the coordinate change

$$\begin{aligned} \bar{z} &= \mathbf{M}\bar{y} \\ d\bar{z} &= \det(\mathbf{M}) dy_1 \cdots dy_m \end{aligned}$$

gives us

$$\begin{aligned} q_{\bar{S}}(\bar{s}) &= \int \cdots \int dy_{m+1} \cdots dy_n \int d\bar{z} \det(\mathbf{M})^{-1} \prod_{i=1}^{m} \delta(z_i - s_i) p_{\bar{X}}(\mathbf{V}\{\mathbf{M}^{-1}\bar{z}, y_{m+1}, \ldots, y_n\}) \\ &= \det(\mathbf{M})^{-1} \int \cdots \int d\bar{\gamma}\, p_{\bar{X}}(\mathbf{V}\{\mathbf{M}^{-1}\bar{s}, \bar{\gamma}\}) \end{aligned}$$

where $\bar{\gamma} = (y_{m+1}, ..., y_n)^T$, and $\{\mathbf{M}^{-1}\bar{s}, \bar{\gamma}\}$ denotes an $n \times 1$ vector.

2.3 The CDF

The cumulative density function for a weighted sum of correlated lognormal variables is

$$\begin{aligned} Q_{\bar{S}}(\bar{k}) &\equiv Prob[S_i \leq k_i, i \in 1, \ldots, m] \\ &= \underbrace{\int_{-\infty}^{k_1} \cdots \int_{-\infty}^{k_m}}_{m} ds_1 \cdots ds_m q_{\bar{S}}(s_1, \ldots, s_m) \\ &= \det(\mathbf{M})^{-1} \int d\bar{s}\, d\bar{\gamma}\, H^{(m)}(\bar{k} - \bar{s})\, p_{\bar{X}}(\mathbf{V}\{\mathbf{M}^{-1}\bar{s}, \bar{\gamma}\}) \\ &= \int d\bar{s}'\, d\bar{\gamma}\, H^{(m)}(\bar{k} - \mathbf{M}\bar{s}')\, p_{\bar{X}}(\mathbf{V}\{\bar{s}', \bar{\gamma}\}) \end{aligned}$$

where $H^{(m)}(\bar{x}) = \prod_{i=1}^{m} H(x_i)$, and $H(x)$ is the Heaviside unit step function, and we substituted $\bar{s}' = \mathbf{M}^{-1}\bar{s}$ in the fourth step.

$$Q_{\bar{S}}(\bar{k}) = \int d\bar{s}'\, d\bar{\gamma}\, H^{(m)}(\bar{k} - \mathbf{M}\bar{s})\, p_{\bar{X}}(\mathbf{V}\{\bar{s}', \bar{\gamma}\})$$

$$= \int d\bar{s}'\, d\bar{\gamma}\, H^{(m)}(\bar{k} - \mathbf{M}\bar{s})\, p_{\bar{X}}(\mathbf{V}\{\bar{s}', \bar{\gamma}\})$$

3 Numerical evaluation of the CDF $Q_{\bar{S}}(k)$

3.1 The Monte Carlo Method

The Monte Carlo method for evaluating an integral

$$I = \int_a^b f(x)dx$$

is based on the observation that

$$I = \sum_{i=1}^{N} \frac{f(X_i)}{N} + \varepsilon$$

where $X_i \in U[a,b]$ is i.i.d uniform random variables in the range $[a,b]$, and the error ε is normal distributed (due to the Central Limit Theorem) with mean 0, and standard deviation proportional to $\frac{1}{\sqrt{N}}$.

In the case we must integrate over the entire $\mathcal{R}^n$, and our integrand is the product of a function $f(\bar{x})$ and the density function $p_{\bar{X}}(\bar{x})$, we can write

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} f(\bar{x}) p_{\bar{X}}(\bar{x}) d\bar{x} \\
&= \int_{-\infty}^{\infty} f(\bar{x}) \prod_{i=1}^{n} \phi(\{\underbrace{\sigma^{-1}(\log \bar{x} - \bar{\mu})}_{=\bar{y}}\}_i) d\bar{x} \\
&= \int_{-\infty}^{\infty} f(\exp(\sigma\bar{y} + \bar{\mu})) \det(\sigma) \prod_{i=1}^{n} \underbrace{\phi(y_i)dy_i}_{=dz_i} \\
&= \int_{[0,1]^n} f(\exp(\sigma\Phi^{-1}(\bar{z}) + \bar{\mu})) \det(\sigma) d\bar{z} \\
&\approx \sum_{j=1}^{N} \frac{1}{N} f(\exp(\sigma\Phi^{-1}(\bar{Z}_j) + \bar{\mu})
\end{aligned}$$

with substitution $\bar{y} = \sigma^{-1}(\log \bar{x} - \bar{\mu})$ in step three and substitution $dz_i = \phi(y_i)dy_i$ in step four. Note that $z_i = \int_{-\infty}^{y_i} \phi(t)dt = \Phi(y_i)$ is the CDF of the normal distribution, and that $\Phi(x)^{-1}$ exists since $\Phi(x)$ is monotone since $\phi(x)$ is strictly positive, and that $\Phi^{-1}(\bar{x}) \equiv (\Phi^{-1}(x_1), ..., \Phi^{-1}(x_n))^T$.

Combining the two above results, we get

$$\begin{aligned} Q_{\bar{S}}(\bar{k}) &= \int d\bar{s}'\, d\bar{\gamma}\, H^{(m)}(\bar{k} - \mathbf{M}\bar{s}')\, p_{\bar{X}}(V\{\bar{s}', \bar{\gamma}\}) \\ &= \int d\bar{w}\, H^{(m)}(\bar{k} - \mathbf{M}\{\mathbf{V}^{-1}\bar{w}\}_{1\ldots m})\, p_{\bar{X}}(\bar{w}) \\ &\approx \sum_{j=1}^{N} \frac{1}{N} \prod_{i=1}^{m} H\left(k_i - \left\{\mathbf{M}\{\mathbf{V}^{-1} \exp(\sigma\Phi^{-1}(\bar{Z}_j) + \bar{\mu})\}_{1\ldots m}\right\}_i\right) \\ &= \sum_{j=1}^{N} \frac{1}{N} \prod_{i=1}^{m} H\left(k_i - \left\{\mathbf{W} \exp(\sigma\Phi^{-1}(\bar{Z}_j) + \bar{\mu})\right\}_i\right) \end{aligned}$$

where, as above, $\mu_i = -\frac{1}{2}(\sum_{j=1}^{n} \sigma_{ij}^2)t$, and where we substituted $\bar{w} = \mathbf{V}\{\bar{s}', \bar{\gamma}'\}$ in step 2, recalled that $\det V = 1$, and denote $\{\mathbf{V}^{-1}\bar{w}\}_{1\ldots m} = (\{\mathbf{V}^{-1}\bar{w}\}_1, ..., \{\mathbf{V}^{-1}\bar{w}\}_m)^T$. In the final step we recall the definition of the operator $\mathbf{M}$ to simplify $\mathbf{M}\mathbf{V}^{-1} = \mathbf{W}$.

Since if Z_{ij} are i.i.d. $U[0,1]$, then all $\Phi^{-1}(Z_{ij})$ are i.i.d. $N(0,1)$. One way to speed up the simulation is to tabulate $\Phi^{-1}(x)$ and interpolate. Another way is to generate independent standard normal distributed numbers $Y = (y_1, y_2)^T$ from $Z = (z_1, z_2)$, i.i.d. $z_i \in U[0,1]$ numbers by using the Box-Muller method, see for instance Numerical Recipes [8].

3.2 Absolute error bound

Suppose the true value of $Q_{\bar{S}}(\bar{k})$ is p and that $x_j = \prod H(...)$ is the value of our integrand in MC round j. Note that x_j are i.i.d.

$$x_j = \begin{cases} 1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-p \end{cases}$$

$E[x_j] = p$, $E[x_j^2] = p$ and $E[x_i x_j] = p^2$ for $i \neq j$. The error is $\varepsilon = \frac{1}{N}\sum_{j=1}^{N} x_j - p$, and $E[\varepsilon] = 0$. The error variance is

$$\begin{aligned} \hat{\sigma}^2 = E[\varepsilon^2] &= E[\frac{1}{N^2}\sum_{i,j} x_i x_j] - 2p\frac{E[x_j]}{N} + p^2 \\ &= \frac{1}{N^2}\sum_{j} E[x_j^2] + \frac{1}{N^2}\sum_{i\neq j} E[x_i x_j] - p^2 \\ &= \frac{p}{N} + \frac{N(N-1)p^2}{N^2} - p^2 \end{aligned}$$

$$= \frac{p - p^2}{N}$$
$$= \frac{Var[x_j]}{N}$$

The probability that the absolute error is less than TOL is

$$Pr[-\frac{TOL}{\hat{\sigma}} < \frac{\varepsilon}{\hat{\sigma}} < \frac{TOL}{\hat{\sigma}}] = 2\Phi(\frac{TOL}{\hat{\sigma}}) - 1$$

because $\varepsilon \in N(0, \hat{\sigma})$ by the CLT. To achieve an absolute error $|\varepsilon| < TOL$ with confidence c, we see that

$$\begin{aligned} \Phi(\frac{TOL}{\hat{\sigma}}) &> \frac{c-1}{2} \\ N &> \frac{Var\{x_j\}}{TOL^2}\left(\Phi^{-1}(\frac{c-1}{2})\right)^2 \end{aligned}$$

Therefore, one must perform $N \approx 4\frac{Var[x_j]}{TOL^2}$ Monte Carlo rounds for $c = 95\%$. To make the algorithm adaptive, $Var[x_j] = p - p^2$ (and thereby N) is estimated during the Monte Carlo simulation.

3.3 Relative error bound

The relative error is $e = \varepsilon/p$, and therefore $N_{rel} = \frac{N}{p^2}$ Monte Carlo rounds must be made to achieve the relative error, which gives the answer with $-\log_{10} e$ significant decimal digits. Note that the number of rounds increases as the square of $(1/p)$, therefore the algorithm converges very slowly for small values of p.

4 Discussion

We have derived the PDF for a weighted sum of correlated lognormal variables $\bar{X}$, and given an adaptive Monte Carlo method for evaluating the CDF $Q_{\bar{S}}(\bar{k}) = Pr[\mathbf{W}\bar{X} \leq \bar{k}] = p$. The Monte Carlo method is used since we are interested in evaluating the function when the number of variables is greater than 20.

The Monte Carlo Method can compute in constant memory, and it scales well in time as long as one only requires a bound on the absolute error. To compute the CDF for 30 lognormal variables and 10 bounding constraints to an absolute error of 0.01 takes around one second on a normal PC workstation

(Intel Pentium, 800MHz). The relative error of the Monte Carlo method given here is $O(1/\sqrt{N}p^2)$ which means that it converges slowly for small p. In our motivating application for this work, we are only interested in a weighted sum of many CDFs of the form described in this paper. Therefore we do not require a low relative error, but only a low absolute error, which the Monte Carlo method handles sufficiently well.

In contrast, most quadrature methods use at least two points in each dimension, and therefore require $O(n^d)$ evaluations. They are difficult to use on functions which are almost zero everywhere except for in a small region, unless one is able to find this region. A high dimensional probability function has this behavior. Stroud [11] provides some quadrature schemes that uses less than two internal points per dimension, but the numerical quadrature still suffers from the problem of finding the sufficient region over which to integrate.

Since we are interested in the CDF of a sum, which is a convolution of the PDFs, a Fourier transform based method would seem appropriate. However, Fourier transform in many dimensions is achieved by applying the transform to one dimension at a time, which means that the time and memory complexity is greater than $O(n^d)$ where d is the number of dimensions, and n is the number of function evaluations in each dimension. This means that the method is infeasible for $d > 30$ on a standard computer. The method is however still interesting for lower dimension problems.

Quadrature methods use too much memory, and their execution time scales exponentially in the number of variables. Fourier transform methods also appear to suffer from the curse of high dimensionality, although these alternative methods may be useful for lower dimension problems, especially when one needs to evaluate the CDF in many points.

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